Chapter 1

Foundations

1.1 Prerequisites

It is assumed that you have met the following functions: sin, cos, tan, ln and the exponential function *exp* which simply takes the number and raises e to that number. You do **not** need to understand these functions to understand the material below, you simply need to know that they are functions.

1.2 Functions

A function is a set of instructions which contain no loopholes (in other words, no ambiguities). Given these instructions and the raw materials, you will always produce the same finished product. There are a large number of ways to describe a set of instructions, the most obvious is the use of language. Another term which we will use in reference to functions is the term *maps to*. When we say that *the function maps* 3 *to* 9*,* we mean that with 3 as its raw materials, the function will produce a 9.

Knowing the result of a function for one input, tells us very little about how the function worked with other numbers. Knowing for instance that the function takes the number 3 and returns the number 9 tells us little about what the function would do to the number 2. In order to obtain the needed generality, we will us a variable, namely a letter which we will treat as if it were a number. Knowing that the function takes the number x and returns the number $\frac{x^3}{3}$ is much more enlightening than knowing simply that 3 maps to a 9. Once we see the result of giving the function an x, we can take apart the instructions that are in the function (in this case, cube the number and then divide by 3). Being able to decode these instructions is essential in learning how to work with functions (for example, it is incorrect to decode the above function as divide by 3 and then cube). Decoding these instructions carefully will be discussed later in this paper, or perhaps in one to follow.

One thing about functions is that the input is completely consumed by the function, this means that once the function has taken an input, you often cannot recover the input from the function. In the case of a simple function (such as a function which simply adds one to all its inputs), you can determine a general method for working backwards from the output to discover what the original input must have been. However, this is not generally the case, as is evident by the fact that the squaring function will produce identical results for the inputs of $+1$ and -1.

I will often refer to a function as an animal which eats its input and leaves behind function droppings (note the term "consumed" in the above paragraph).

Functional Notation

Frequently, we will want to make repeated uses of the same function and so rather than referring to a function as "that function which cubes the number and then divides by 3" we will give the function a short name. Function names (by convention) tend to be f, g and h. When using a function's name, we will use the notation $f(x)$ to indicate the finished product of f when it is given the number x as input. This notation is referred to as *functional notation.* For example, if f refers to the function cube and divide by three, then $f(3) = 9$, $f(2) = 8/3$ and $f(x) = x^3/3$. The last of these equations is frequently the way that a function will be presented to the reader. For example:

$$g(x) = \left(x^3 - 2x^2 + \sqrt{\frac{x^2 - x}{x^3 - 2x^{17} + 4}} \right)^{x^2+2x+1}$$

describes a function whose description in the English language would almost constitute a term paper. However, the mathematically description makes it possible to compute that $g(0) = 0$.

Notice that the symbols $f(x)$ constitute function droppings and not a function. We can tell a lot about a function from its droppings. The equation $f(x) = x^3/3$ merely tells us that f droppings look like whatever f ate cubed and divided by 3.

One particularly sticky type of instruction for a function (often given in functional notation) is the case statement. These functions leave droppings that look like:

$$f(x) = \begin{cases} x^2 + 1 & x < 0 \\ x - 2 & x \geq 0 \end{cases}$$

The way to follow these instructions is to first decide to which expression your input corresponds. In this case, all negative numbers will correspond to the first expression given $(x^2 + 1)$ and all non-negative numbers will correspond to the second $(x - 2)$. This means that if you are feeding f a -1, you will get $(-1)^2 + 1 = 2$ back. If you are feeding f a $+1$, you will get $1 - 2 = -1$ back. In some cases, you will not be able to determine which of the two expressions you should use (for example, $f(s)$ where you cannot determine the sign of s) and in these cases you should simply leave the result in functional notation.

Variable Notation

Another method of describing a function is simply to provide an equation for which relates the input to the output in some fashion. The example "$y = 3x^2 - 3x + 1$ is in the most common form (one in which y has been isolated). However, there is no requirement that a function needs to have been solved for y, for example the equation $e^{x+y} = x + y + 1$ cannot be easily solved for y, but for every value of x there is a unique value of y which will satisfy the equation and so the equation determines a function (actually, the value of y will always be $-x$, the interesting fact is that this is the only solution which works, in this course, you will learn how to show that this is the case). However, one problem arises with the variable notation which is not evident in the functional notation. Given an equation such as $e^{r+s} = r + s + 1$, there is little indication of which variable is intended to be the input and which is the output. If we are given a value for r, we can get a value for s, similarly, given a value for s, we can get a value for r. As a result we have the following requirement: All functions described as equations must come with indications of which variable is considered an input and which is considered an output. A variable which will be an input will be called a *independent variable*, a variable which will be considered an output will be called a *dependent variable*. For example, given the equation $r = s^3$, if s is the independent and r is the dependent, then this is the cubing function. On the other hand, if r is independent and s is dependent, then this is the cube root function.

In the animal metaphor, a dependent variable is the droppings of the function, the independent variable is a sample of function food. Again, the actual function is nowhere to be seen, but much can be learned by looking at what it left behind.

As usual, mathematicians are lazy and we will often avoid mentioning which variables are dependent and which are independent. The following guidelines often remove ambiguity.

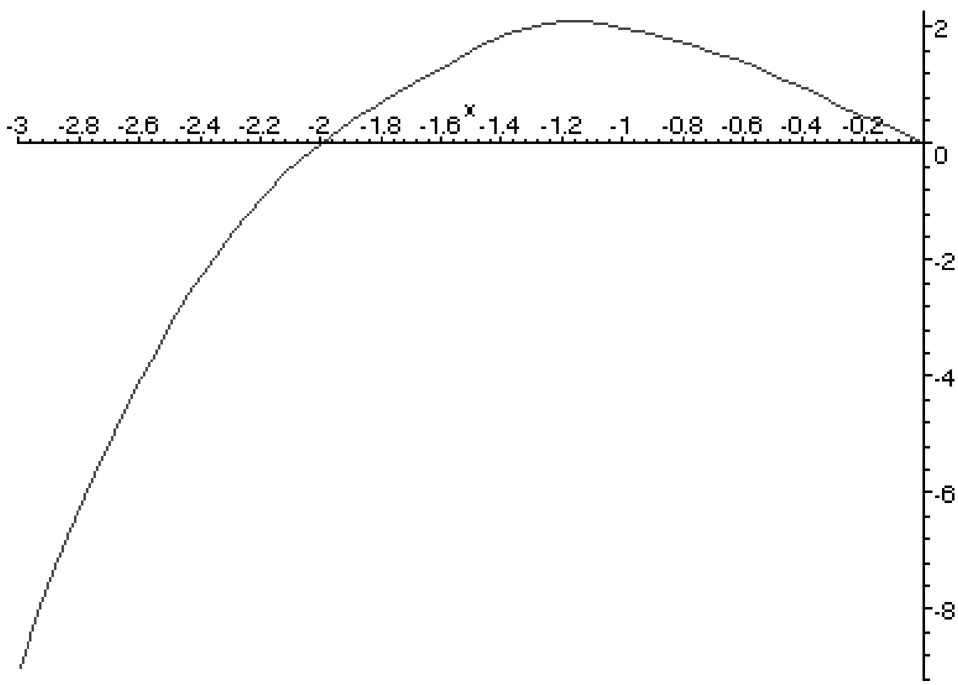
- If one variable is solved for, then it is often a dependent variable and all other variables are independent.
- If the variables are x and y and the problem does not involve changes over time, then x is often independent and y is often dependent.
- If there exists a variable which represents time (either explicitly given as t or based on the context of the problem), then t is the independent variable and all other variables are dependent.

One (in my opinion) rotten notation which has arisen to help keep track of dependent and independent variables is a hybrid of the functional and the variable notations. In this unholy union, a dependent variable is always followed by a parenthesized list of the independent variables. For example, in the equation $r = s^3$ if we intended r to be dependent we would write $r(s) = s^3$ and if we intended s to be dependent we would write $r = (s(r))^3$. I prefer to avoid this notation since if r and s are variables, then $r(s)$ could be interpreted as multiplication of r and s which can cause confusion. This confusion (I hope) won't arise as frequently with the functional notation where $f(x)$ cannot be thought of as multiplication since f is a function and such things are not multiplied by x's.

Graphical Descriptions

The primary problem with the two descriptions above is that they either provide us with a fairly obscure definition such as the one for g given on page 3 or it will provide us only with only a few pin-holes of insight as we plug-in a few values. A graphical description is capable of presenting a much broader view of the function.

For example, the graph of the function g for inputs over the range $[-3, 0]$:



and from this we can get a general overview of how g behaves (the fact that it peaks somewhere around $x = -1$ is not evident in the functional notation).

Sometimes, we will only be given a graph and will be required to determine the function's output values for given input values and this requires that you be able to read a graph. If you are given an input value, you should find that value on the horizontal axis (often the x-axis). Next you should find the intersection point of the vertical line through that point and the graph (if it exists) and the vertical coordinate (often the y-value) is the output of the function. Notice that if there are multiple intersections, then the instructions are ambiguous and the graph does not define a function.

1.3 Domain and Range

The domain of a function is the collection of all numbers which the function will eat (alternatively, it is the collection of all numbers which will not kill the function). For example, the function $f(x) = 1/x$ cannot be fed the number 0 because the instructions "divide 0 into 1" do not result in anything (the result is undefined). The primary things which you will need to be careful with are dividing by 0 and taking an even root of a negative number.

On the other side, the range of a function is the collection of all numbers which could be produced by the function. Using the function f above, every number except 0 can be obtained as a result from f, however, there is no number which when divided into 1 gives 0.

1.4 Common Functions

The course really covers only algebraic functions (described below), however, they are so well behaved that they are rather boring. Although not officially part of the course, I will make mention of some other functions. I will provide you with everything you need to know about these functions as this information is needed. You will probably be surprised at how little you actually need to know about sin, cos, ln and exp before you use them in calculus.

Algebraic Functions

Algebraic Functions come from addition, subtraction, multiplication, division, raising to a rational power and taking a rational root. By rational I mean a number which can be written as $\frac{p}{q}$ for some integers p and q, this means that $f(x) = x^{\frac{314}{100}}$ is algebraic and $f(x) = x^{\pi}$ is not. Algebraic functions are exceptionally nice and well behaved. The only forbidden operations are dividing by zero and taking even roots of negative numbers. Most of this course will focus on algebraic functions.

Trigonometric Functions

The two trigonometric functions which we will use are sin and cos. The following is the formal definition of how to obtain the values $\sin(t)$ and $\cos(t)$ for the number t.

1. Draw a circle in the xy-plane with its center at the origin and a radius of 1.
2. If t is positive, pace counter-clockwise around the circle a distance of t. If t is negative, pace clockwise around the circle a distance of t. If t is zero, you need not move anywhere.
3. Use your PPS (Plane-Positioning-System) to determine your x and y coordinates. $\sin(t)$ is your y-coordinate and $\cos(t)$ is your x-coordinate.

Although you will not have to ever follow these instructions for this course (I will allow you to let your calculator do the dirty work) you should be roughly familiar with these function's definitions.

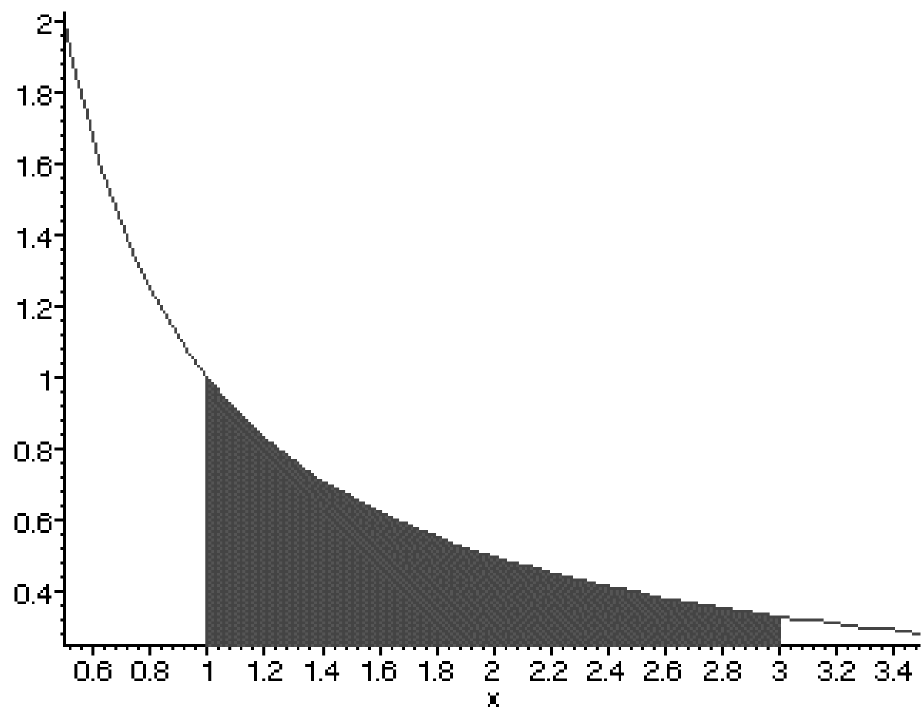
The instructions above can be carried out with any number and so the domain of sin and cos consists of all real numbers. On the other hand, since you are constrained to the unit circle, your x and y coordinates remain between -1 and 1 and so the range of sine and cosine is the interval $[-1, 1]$.

There are inverse functions for sin and cos which are denoted arcsin and arccos. The general idea is that if you feed a position on the unit circle to arcsin and arccos, you can discover how far you need to walk (and in which direction) to get to that location. The only important thing which you need to know about these are: they exist and they are not algebraic.

Logarithms and Exponentials

We will only focus on the natural logarithm for this course (all other logarithms are just constant multiples of this one). The formal definition of $\ln(t)$ for the number t is given by:

1. Draw vertical line from $(1, 0)$ to $(1, 1)$.
2. Draw a horizontal line from $(1, 0)$ to $(t, 0)$.
3. Draw a vertical line from $(t, 0)$ to $(t, 1/t)$.
4. Draw the portion of the curve $y = 1/x$ between the points $(1, 1)$ and $(t, 1/t)$. This makes a funny trapezoidal shape (shaded in the figure for $t = 3$).



5. If $t > 1$, then the area inside the shape is the value of $\ln(t)$. If $t < 1$, then the negative of the area is the value of $\ln(t)$. If $t = 1$, you have drawn a line segment which has zero area so $\ln(1) = 0$.

The domain of ln will be all positive numbers because if t were negative, then you would not have a closed figure when you finished drawing the shape. The range of ln is all numbers, although I won't try to show that here.

The inverse function (which answers the question, "what should t be if I need a specified area?") is the exponential function exp and there exists a number e such that $e^{p/q} = \exp(p/q)$ for all rational numbers. Hence we use the notation e^x to stand for $\exp(x)$.

Now we are finally able to define general exponentiation. Given two numbers x and y, we define x^y to mean $\exp(y \ln(x))$. If y is a rational number, this is the same exponentiation found in the algebraic functions. If y is an irrational number, then this is something new. **DON'T PANIC:** even though this is something new, all the old rules work just like normal and you can continue in blissful ignorance that $f(x) = x^\pi$ is quite a monstrous function.