(ii) If $K \neq 0$, then for $t \neq 0$ [that is, any point (x, y) on the line other than (x_0, y_0)], we have $F(x, y) = Kt^2 \neq 0$. And so (x, y) does not belong to the conic $F(x, y) = 0$. Since (x_0, y_0) is the only point on the line that belongs to the conic, (2) is the equation of the line tangent to the conic section at (x_0, y_0).

Example. The tangent line to the graph of $x^2 - xy + y^2 - 7 = 0$ at $(-1, 2)$ is

$$x^2 - xy + y^2 - 7 = (x + 1)^2 - (x + 1)(y - 2) + (y - 2)^2$$

or $4x - 5y + 14 = 0$ (see Figure 1).

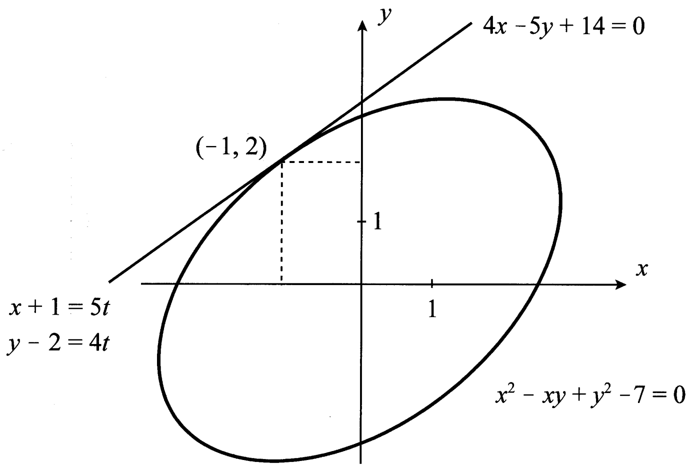


Figure 1.

For related ideas see [**1**], [**2**], [**3**].

References

1. Jorge Arao, Tangent without calculus, *College Mathematics Journal* **31**:5 (2000) 406–407.
2. John Staib, Some pre-calculus algebra, *The Two Year College Mathematics Journal* **10** (1979) 89–95.
3. H. Thurston, Tangents, an elementary survey, *Mathematics Magazine* **41**:1 (1969) 1–11.

A Hairy Parabola

Aaron Montgomery (montgoaa@cwu.edu), Central Washington University, Ellensburg, WA 98926

The Mark Twain boat company offers tours of the Mississippi River. If 50 people sign up, the boat company charges each person \$10. However, for each group of

10 people beyond the first 50, everyone's ticket price decreases by 50 cents. What sized tour group maximizes the company's revenue?

Problems of this type (maximizing yield) frequently occur in first term calculus or even college algebra (where the vertex formula can be used). In many cases, the solution is obtained by a process similar to the following, where x is the groups of 10 beyond the first 50 and R is the revenue

$$R = (50 + 10x)(10 - 0.5x)$$
$$\frac{dR}{dx} = 0 \Longrightarrow x = 7.5$$
$$\left.\frac{d^2R}{dx^2}\right|_{x=7.5} < 0$$

The student's first response might be that there should be $50 + 10(7.5) = 125$ people each paying $10 - 0.5(7.5) = 6.25$ dollars for a revenue of 781.25 dollars. When pressed on the correctness of the answer, some students will adjust it by stating that x needs to be an integer because discounts come only for complete groups of 10. This means that either $x = 7$ and there is a maximum revenue of $120 \times 6.50 = 780$ dollars or $x = 8$ and there is a maximum revenue of $130 \times 6.00 = 780$ dollars. In either case the maximum revenue is 780 dollars.

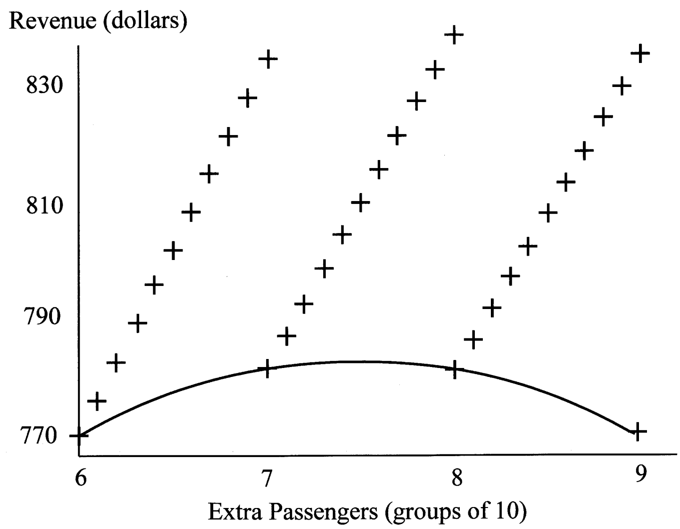
Other students might notice that x can actually be 7.5, but that the ticket price would be \$6.50 (and not \$6.25). This means the actual revenue will be higher: $125 \times 6.50 = 812.50$ dollars. Asking students to generate a table based on the number of people and the ticket prices might help convince students that something strange is happening.

number of people	price of ticket (dollars)	revenue (dollars)
119	7.00	833
120	6.50	780
⋮	6.50	⋮
124	6.50	806
125	6.50	812.50
126	6.50	819
⋮	6.50	⋮
129	6.50	838.50
130	6.00	780

From this (or independently) students can usually see that passengers 121 through 129 increase the revenue because they add their ticket price to the group without forcing a price reduction. This means that the maximum revenue, \$838.50, is achieved with 129 passengers.

The problem is that the discrete situation being modeled and the continuous model are dissimilar in a way that requires careful interpretation of the results. This can be seen in the following graph (a hairy parabola), where the crosses graph the discrete function and the curve represents the continuous function.

Problems like this are particularly important in modern curricula. The actual steps to produce a maximum value for the continuous model are straightforward and can even



be presented in algebra courses, using the vertex formula for a parabola. However, the interpretation of the numerical results requires students to think about the actual situation and relate their "mathematical" solutions to the "real world" situation.

———∘———

An Improved Remainder Estimate for Use with the Integral Test

Roger B. Nelsen (nelsen@lclark.edu), Lewis & Clark College, Portland, OR 97219

Nearly every modern calculus text contains the following result in the chapter on infinite series: If $\sum_{i=0}^{\infty} f(i)$ converges to S by the integral test, and $S_n = \sum_{i=1}^{n} f(i)$ denotes the nth partial sum of the series, then the "remainder" $R_n = S - S_n = \sum_{i=n+1}^{\infty} f(i)$ satisfies

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx. \tag{1}$$

The hypotheses for the integral test require that f be continuous, positive, and decreasing on $[1, \infty)$. In [**1**], R. K. Morley showed that if, as is often the case, f is also *convex* (concave up) on $[1, \infty)$, then the "traditional" estimate (1) can be improved to

$$\int_{n}^{\infty} f(x)\,dx - \frac{1}{2}f(n) \le R_n \le \int_{n}^{\infty} f(x)\,dx - \frac{1}{2}f(n+1). \tag{2}$$

The purpose of this Capsule is to note that, under the same hypotheses, these estimates can be further sharpened to

$$\frac{1}{2}f(n+1) + \int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n+1/2}^{\infty} f(x)\,dx. \tag{3}$$

The proof of (3) follows directly from the observation that for convex functions, the midpoint rule underestimates integrals while the trapezoidal rule overestimates