Strengthening Student Conceptions of Function

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1. Background

Student's misconceptions of function has been noted by numerous researchers as is evidenced by [13]. The material proposed here are designed to aid in addressing misconceptions related to composition, inversion and graphing.

The work of [5] and [7] conclude that student difficulties "could be attributed to student difficulties with composition and decomposition of functions" and that students "need to move from a focus on "inside/outside" structure based on parentheses to a perception of composition of functions" respectively. However neither article provides a plan to remediate this problem. The function diagrams (described briefly below) were developed as an attempt to address this concern.

The work of [8] suggest that students perceive the current graphing curricula as "[acting] on *equations* rather than functions, and then simply learn the connection between transformations of equations and transformation of graphs." To remediate this problem, [8] suggests an alternative curricula based on *ISETL* which appears to remediate some of these problems and [11] suggests a solution through specialized software. More recently [3] also suggests the use of specialized software to address this problem. The graphing technique described below does share some features with the previous work but does not require computer activities which may make it more suitable in those situations where computer projects may be problematic. This is not to say that students should be presented with one technique or the other. The author has found that using computer techniques *alongside* the new graphing technique has been effective, with each technique supporting the other.

A common thread in both of the above problems is that students are able to reduce the rich concept of function to the much more limited concept (that of variable expression or curve). In the algebraic case, this reduction is so strong that [17] reports that students presented with a malformed definition of the function f given by "$f(n) = x^2$" believe that f has been defined as the squaring function. Similar problems arise in situations where graphs are interpreted literally without attention paid to the meaning of the axes as demonstrated in [4]. The investigator believes that, to some extent, these problems are ones that we have brought upon ourselves. The standard curriculum traditionally presents function definitions in a standardized form that allows students to ignore the deeper concept and get by with the weaker understanding that fails them later.

2. The Proposed Materials

The materials are centered around something referred to as function diagrams. The primary goal in this section is to contrast the traditional approach with the new approach. We will present sample questions, a traditional solution and the proposed solution. A more detailed description of the experimental solutions can be found at [14]. The traditional methods of solution presented below are taken from current pre-calculus texts, all of which contain the word "function" in their title.

Algebraic Reduction. In an attempt to avoid having students reduce the function definition to "the complicated expression," students can be asked the following:

Q: Given the function f defined by $f(x^3) = x^2 + 1$, what is $f(u)$?

A traditional solution might reason as follows:

$$u = x^3 \implies x = \sqrt[3]{u} \text{ and so } f(u) = x^2 + 1 = \sqrt[3]{u}^2 + 1$$

The experimental technique reduces f to a sequence of arrows, makes a detour through x so we can identify each "basic" step, changes the input and computes the output through the sequence of steps described by the arrows.

$$x^3 \xrightarrow{\qquad\qquad\qquad f \qquad\qquad\qquad} x^2 + 1$$

$$x^3 \xrightarrow{\sqrt[3]{\ }} x \xrightarrow{\hat{}2} x^2 \xrightarrow{+1} x^2 + 1$$

$$u \xrightarrow{\sqrt[3]{\ }} \sqrt[3]{u} \xrightarrow{\hat{}2} \sqrt[3]{u}^2 \xrightarrow{+1} \sqrt[3]{u}^2 + 1$$

As an added advantage, the proposed solution allows the student to answer the question "what is $f(x)$?" as easily as the given question. The traditional method breaks down because the solution of $x = x^3$ leads nowhere.

Function Decomposition. To strengthen student ability with the decomposition of functions, students can be asked questions of the following type:

Q: The function f is given by $f(x) = \sin(\sqrt{x^3 + 1})$, determine two function g and h such that neither g nor h is the identity and $f = g \circ h$.

Under the traditional curricula, students are trained to recognize an "inside" $h(x) = x^3 + 1$ and told that the remainder of the expression is g and so $g(x) = \sin(\sqrt{x})$. Again, students are asked to focus on the structure of the given algebraic expression. The experimental approach decompose the function f completely:

$$x \xrightarrow{\hat{}3} x^3 \xrightarrow{+1} x^3 + 1 \xrightarrow{\sqrt{\ }} \sqrt{x^3 + 1} \xrightarrow{\sin} \sin(\sqrt{x^3 + 1})$$

$$x \xrightarrow{\qquad\quad h \qquad\quad} x^3 + 1$$

$$x \xrightarrow{\qquad\qquad\quad g \qquad\qquad\quad} \sin(\sqrt{x})$$

Then students can split the sequence at any point, the split matching that described by the traditional method is given above. Notice that the split here may not be the most appropriate (for example, if the student were attempting to use the chain rule). By laying bare the complete structure, the student has the ability to see the alternatives open to them. The material also provides students with the ability to diagram the four binary operators $(+, -, \times, /)$, which allows the same type of question to be asked in cases where f is not simply a composition.

Inverse Functions. Inverse functions also present a use for function diagrams. For example,

Q: The function f is defined by $f(x) = \sqrt{x^3 - 1}$, determine $f^{-1}(x)$.

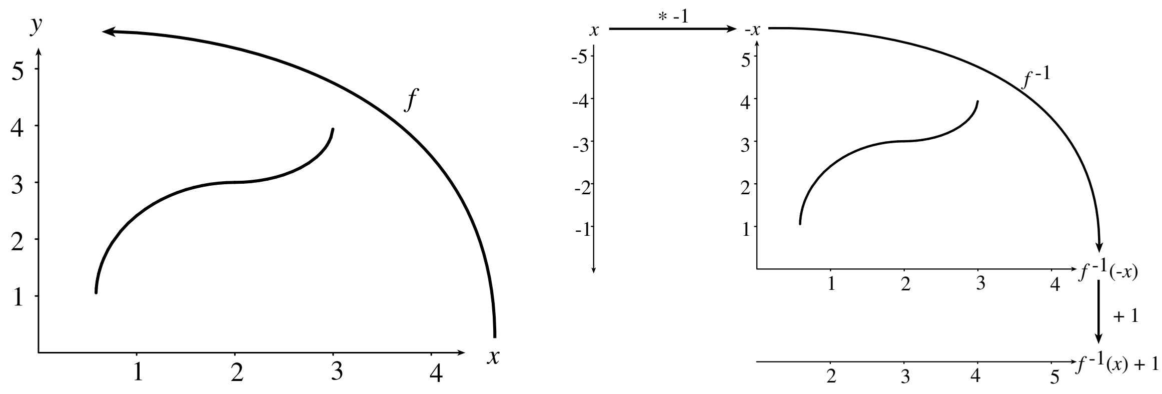

FIGURE 1. Graphing with Arrows

A traditional approach would be to manipulate the following equations.

$$y = \sqrt{x^3 - 1}$$
$$x = \sqrt{y^3 - 1}$$
$$\sqrt[3]{x^2 + 1} = y$$

Under the experimental approach, students are encouraged to simply reverse the arrows, place x on the far right and then follow the arrows to the left.

$$x \xrightarrow{\;\hat{}\,3\;} x^3 \xrightarrow{+(-1)} x^3 - 1 \xrightarrow{\sqrt{\;}} \sqrt{x^3 - 1}$$

$$\sqrt[3]{x^2 + 1} \xleftarrow{\sqrt[3]{\;}} x^2 + 1 \xleftarrow{+1} x^2 \xleftarrow{\;\hat{}\,2\;} x$$

Notice that the diagram in the solution has arrows pointing in the "wrong direction." Reasons for this can be found in [14]. The investigator is aware that this technique has begun to appear in some texts (in particular, [10]). However, this technique is only introduced after students have been trained to find inverses by solving equations. It is the investigator's belief that students should be introduced to the topic using the experimental approach and then shown the technique using equations after the concepts have been learned.

Graphing. Graphing is another area where the function diagrams can be used to aid understanding.

Q: Given the graph of $y = f(x)$ on the left in Figure 1, draw the graph of $y = f^{-1}(-x) + 1$.

Using traditional techniques, students are told to reflect the given graph around $y = x$, reflect the resulting graph around the y-axis and raise it by 1 unit. The (almost final) solution proposed by the investigator is given in the graph to the right in Figure 1. The solution starts with the graph of $y = f(x)$, changes the direction of the arrow, inserts $\xrightarrow{\times -1}$ at the start and $\xrightarrow{+1}$ at the end of the arrow sequence. After this the tick marks on the new axes are determined (by applying the appropriate function or its inverse based on the direction of the arrows). The result is the graph shown above. Students would be expected to redraw the curve with more standard axis placement and orientation.

Graphical Reduction. One problem discussed above was the students attempt to reduce a function to a curve. However, this reduction becomes more difficult with the graphing technique described in the previous example. The curve presented in Figure 1 describes the function f as well as any function of the form $x \to af(bx+c)+d$ or $x \to af^{-1}(bx+c)+d$. The student is required to carefully examine the direction of the arrows (or the labelling of the axes) to decide which function is being described.

Conclusion. While the above examples have been chosen to suggest the advantages of the experimental techniques, it is certainly true that there are places where the traditional techniques have their advantages. For example, inverting $x \to (x+1)/(x-3)$ is probably easier done by solving

$$y = \frac{x+1}{x-3}$$

for x than by writing f as a sequence of simple steps (requiring students to do polynomial division). Similarly, the experimental graphing technique is an overly complicated method to determine the graph of $y = f(x) + 2$ from a graph of $y = f(x)$. The proposed materials hope to first develop a student understanding of function and then provide them with alternative methods of computation (relying on solving equations or geometric transformations) when appropriate.

3. Evaluation of the Materials

If funded, this project will support an effort to document the effectiveness of these techniques.

Theoretical Framework. There are a variety of descriptions about how the concept of function develops. Whether we use the terminology of "structural and operational" found in [15], "action, process, and object" found in [2] or "process and concept" found in [12], it is hypothesized that students move from a phase where they see a function as a dynamic process which acts on numbers to a static object which can itself be acted upon. In this particular study, the first goal will be to determine if the subject has a process level understanding of the function concept. Even though this level of understanding is insufficient for a discussion of the composition operator $(f,g) \to f \circ g$ or the inversion operator $f \to f^{-1}$, it is sufficient to discuss composition and inversion, [18].

The process and object development path provides a description of a vertical development of the function concept and can be used to design problems and tasks of varying difficulty. A second aspect of this particular study is the horizontal development of the function concept. Previous studies have tried to determined by the student's ability to move between the various representations of function (see [1, 9] for example). Recognition that two different representations are describing the same underlying concept as well as the ability to move between these representations can be used to indicate whether a subject has recognized that these representations are tied together. In the terminology of [5], this would indicate a transition from the Intra stage of a scheme to the Inter or Trans stages. The investigator believes that such a transition would indicate that a subject had begun to develop a conceptual understanding of function rather than facility at each representation individually. In the terminology of [16], this would indicate that the subject was using an appropriate cognitive root for the function concept.

The Experiment. Two parallel sequences of precalculus will be offered in Fall 2004–Winter 2005 terms. During this time, once course will be taught using the proposed materials and the other course will be taught using the standard precalculus material (currently [6]). Students in both classes will be presented some common questions on their final examinations allowing a quantitative comparison of the two methods.

During the Spring 2005 term, students who completed the precalculus sequence in Winter 2004 would be interviewed. The goal of these interviews is to determine the effectiveness of the proposed material. Interviews will focus on first establishing the subject's level of understanding using tasks in the traditional representations (algebraic, graphical and numeric). Once this has been done, the investigator will attempt to determine the students ability to transfer information given in one representation to a different representation or to transfer information given in an unfamiliar representation into one of the traditional representations (for example, asking a student to generate a graph describing a situation arising in a physical or verbal model which is unfamiliar to the subject). The interview will conclude with a few questions that attempt to determine the ability of the student to treat a function as an object. While none of the material to be studied in this project is directly designed to encourage this transition, it is possible that manipulating the arrow diagrams will increase the students ability to treat a function as an entity. There is a discussion of using function diagrams to encourage this transition in calculus courses can be found in [14].

The Hypothesis. The investigator hypothesizes that more students from the experimental curriculum will have reached a process level understanding of composition, inversion and graphing. Furthermore, they will have an increased ability to transfer information between the algebraic and graphical representations.

4. Budget Justification

The budget requests release time for the investigator to conduct, transcribe and analyze interviews with students who are using the materials during the 2004–2005 Academic Year. The one month of salary will allow the investigator a one-course reduction in teaching load for the Spring 2005 quarter to allow time for interviewing. Funds are also being requested to allow the investigator to pay each participating student (up to a maximum of 50) a $10 stipend.

5. Investigator Information

Dr. Aaron Montgomery is an Assistant Professor of Mathematics at Central Washington University. Prior to this, he was a Visiting Assistant Professor at Purdue University North Central and has been involved in a number of curriculum projects. During Summer 1998 and Academic Year 1998–1999, he was a co-recipient of a grant from the Indiana Higher Education Telecommunication System to develop an online statistics course at Purdue University North Central. In Summer 1999, he analyzed the current state of the developmental mathematics sequence at Purdue University North Central and developed materials for the Developmental Algebra courses that are still in use there. In Fall 1999 he joined Research in Undergraduate Mathematics Education Community (RUMEC), and is currently involved in the RUMEC linear algebra project (both as author and as a researcher assessing this materials' effectiveness). In Spring 2000 he received a mini-grant from the MAA for the study of student's concepts of limits. It was this mini-grant that laid the groundwork for the preliminary material described in this proposal.

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Budget

	FOR NSF USE ONLY					
ORGANIZATION Central Washington University	PROPOSAL NO.			DURATION		
					Proposed	Granted
PRINCIPAL INVESTIGATOR/PROJECT DIRECTOR Aaron Montgomery	AWARD NO.					
A. SENIOR PERSONNEL: PI/PD, Co-PI´S, Faculty and Other Senior	NSF Funded Person Months			Funds Requested	Matching Funds	Total Project Costs
	CAL	ACAD	SUMR			
1 Aaron G Montgomery, Assistant Professor	0.00	1.00	0.00	4,921.34		4,921.34
(1) TOTAL SENIOR PERSONNEL						
B. OTHER PERSONNEL (SHOW NUMBERS IN BRACKETS)						
1. (0) POST DOCTORAL ASSOCIATES						
2. (0) OTHER PROFESSIONALS						
3. (0) GRADUATE STUDENTS						
4. (0) UNDERGRADUATE STUDENTS						
5. (0) SECRETARIAL - CLERICAL (IF CHARGED DIRECTLY)						
6. (0) OTHER						
TOTAL SALARIES AND WAGES (A+B)				4,921.34		4,921.34
C. FRINGE BENEFITS (IF CHARGED AS DIRECT COSTS)				1,476.40		1,476.40
TOTAL SALARIES, WAGES AND FRINGE BENEFITS (A+B+C)				6,397.74		6,397.74
D. PERMANENT EQUIPMENT						
TOTAL EQUIPMENT						
E. TRAVEL 1. DOMESTIC (INCL. CANADA, MEXICO AND U.S. POSSESSIONS)						
2. FOREIGN						
F. PARTICIPANT SUPPORT COSTS						
1. STIPENDS 50 stipends at $10 each						
2. TRAVEL						
3. SUBSISTENCE						
4. OTHER						
(50) TOTAL NUMBER OF PARTICIPANTS				500.00		500.00
G. OTHER DIRECT COSTS						
1. MATERIALS AND SUPPLIES						
2. PUBLICATION COSTS/DOCUMENTATION/DISSEMINATION						
3. CONSULTANT SERVICES						
4. COMPUTERS SERVICES						
5. SUBAWARDS						
6. OTHER						
TOTAL OTHER DIRECT COSTS						
H. TOTAL DIRECT COSTS (A THROUGH G)				6,897.74		6,897.74
I. INDIRECT COSTS (SPECIFY RATE AND BASE)						
TOTAL INDIRECT COSTS						
J. TOTAL DIRECT AND INDIRECT COSTS (H+I)						
K. RESIDUAL FUNDS						
L. AMOUNT OF THIS REQUEST (J) OR (J MINUS K)				6,897.74		6,897.74
M. COST SHARING: PROPOSED LEVEL						
PI/PD TYPED NAME & SIGNATURE*	FOR NSF USE ONLY					
	INDIRECT COST RATE VERIFICATION					
INST. REP. TYPED NAME & SIGNATURE*	Date Checked		Date of Rate Sheet		Initials-DGA/Program	

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